

CIE A-Level 數學 9709 Paper 6

A-Level 數學 | 時限：75 分鐘 | 總分：75 分 | 共 9 題

甲部 概率分佈與連續隨機變數 (40 分)

全部題目必須作答，須展示步驟並註明所用公式；答案須準確至三位有效數字。

1. A helpline receives calls at random at an average rate of 3.2 calls per hour. The number of calls in any one hour is modelled by a Poisson distribution. (a) Find the probability that exactly 5 calls arrive in a randomly chosen hour. (b) Find the probability that at most 2 calls arrive in an hour. (c) Find the probability that at least 6 calls arrive in an hour.

答案：(a) $X \sim \text{Po}(3.2)$, so $P(X = 5) = e^{-3.2} \times 3.2^5 \div 5! = 0.1140$. (b) $P(X \leq 2) = e^{-3.2}(1 + 3.2 + 3.2^2 \div 2) = 0.3799$. (c) $P(X \geq 6) = 1 - P(X \leq 5) = 1 - 0.8946 = 0.1054$. The Poisson model requires calls to occur singly, independently and at a constant average rate. [3, 3, 2]

2. A large box contains 400 components. The probability that any one component is defective is 0.012, independently of the others. (a) Write down the exact distribution of the number of defectives and state why a Poisson approximation is appropriate. (b) Use a Poisson approximation to find the probability that a box contains no defective component. (c) Estimate the probability that a box contains at least 5 defective components.

答案：(a) $X \sim B(400, 0.012)$ exactly; $n = 400$ is large and $p = 0.012$ is small with $np = 4.8$, so X is approximately $\text{Po}(4.8)$. (b) $P(X = 0) = e^{-4.8} = 0.0082$. (c) $P(X \geq 5) = 1 - P(X \leq 4) = 1 - 0.4763 = 0.5237$. [2, 3, 3]

3. A continuous random variable X has probability density function $f(x) = k(4 - x)$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise. (a) Show that $k = 1/8$. (b) Find $E(X)$ and $\text{Var}(X)$. (c) Find $P(X \text{ less than } 2)$ and explain whether the median of X is greater than 2.

答案：(a) The area is $k \int_0^4 (4 - x)dx = k(16 - 8) = 8k$, so $k = 1/8$. (b) $E(X) = (1/8) \int_0^4 (4x - x^2)dx = (1/8)(32 - 64/3) = 1.3333$; $E(X^2) = (1/8)(256/3 - 64) = 2.6667$, so $\text{Var}(X) = 0.8889$ ($8/9$). (c) $P(X \text{ less than } 2) = (1/8)(8 - 2) = 0.75$, which is less than 0.5, so the median lies above 2 because the density decreases with x . [2, 3, 3]

4. The waiting time T , in minutes, between consecutive customers at a counter is modelled by an exponential distribution with mean 5 minutes. (a) Find the probability that a customer waits more than 8 minutes. (b) Find the probability that a waiting time is between 2 and 6 minutes. (c) Find the median waiting time.

答案：The density is $f(t) = 0.2e^{-0.2t}$. (a) $P(T > 8) = e^{-1.6} = 0.2019$. (b) $P(2 < T < 6) = e^{-0.4} - e^{-1.2} = 0.3691$. (c) Solve $1 - e^{-0.2m} = 0.5$, so $m = 5 \ln 2 = 3.4657$ minutes. [3, 2, 3]

5. The masses of apples of variety A are normally distributed with mean 20 g and standard deviation 3 g. Variety B has masses normally distributed with mean 15 g and standard deviation 2 g. Two apples are chosen independently, one of each variety. (a) Find the probability that their total mass exceeds 40 g. (b) Find the probability that the mass of the A apple is less than the mass of the B apple plus 3 g. (c) Find the probability that $3A - 2B$ exceeds 35 g.

答案：(a) $A + B \sim N(35, 13)$, sd 3.6056; $z = (40 - 35)/3.6056 = 1.3868$, so $P = 0.0828$. (b) $A - B \sim N(5, 13)$; $z = (3 - 5)/3.6056 = -0.5547$, so $P = 0.2895$. (c) $3A - 2B \sim N(30, 97)$, sd 9.8489; $z = 0.5077$, so $P = 0.3058$. [3, 3, 2]

乙部 統計推斷與假設檢定 (35 分)

全部題目必須作答；假設檢定須寫出 H_0 、 H_1 、檢定統計量及結論。

6. A random sample of 10 packets from a production line gave the following masses, in grams: 12.4, 11.8, 12.9, 12.1, 13.0, 12.6, 11.9, 12.3, 12.7, 12.5. (a) Calculate unbiased estimates of the population mean and variance. (b) State the estimate of the standard error of the sample mean if the population variance is known to be 0.16. (c) Hence find a 95% confidence interval for the population mean.

答案：(a) $\bar{x} = 124.2 \div 10 = 12.42$ g; $\sum(x - \bar{x})^2 = 1.4560$, so $s^2 = 1.4560 \div 9 = 0.16178$ and $s = 0.4022$ g. (b) $se = 0.4 \div \sqrt{10} = 0.1265$. (c) $12.42 \pm 1.96(0.1265) = 12.42 \pm 0.2479$, giving (12.172, 12.668) g. [3, 3, 2]

7. Over a long period the number of faults reported on a machine averaged 2.0 per hour. After a service, a random sample of 30 one-hour periods contained 45 faults in total. Test at the 5% significance level whether the mean rate has fallen, assuming the number of faults per hour still has a Poisson distribution. State the distribution used and give the p-value.

答案： $H_0 : \lambda = 2.0$ per hour; $H_1 : \lambda < 2.0$. Under H_0 the total in 30 hours is $Po(60)$. The p-value is $P(X \leq 45) = 0.0266$. The normal approximation with continuity correction gives $z = (45 + 0.5 - 60) \div \sqrt{60} = -1.8719$ and $\Phi(z) = 0.0306$. Since the p-value is less than 0.05, reject H_0 : there is evidence that the fault rate has fallen. [3, 3, 3]

8. A machine fills bottles with a volume that is normally distributed with variance 36 ml^2 . A sample of 25 bottles has mean volume 52.4 ml. It is claimed that the mean volume is 50 ml. (a) Test the claim against the alternative that the mean exceeds 50 ml, using a 5% significance level. (b) Find the critical region in terms of the sample mean. (c) Find the probability of a Type II error when the true mean is 53 ml.

答案：(a) $H_0 : \mu = 50$; $H_1 : \mu > 50$. $se = 6 \div \sqrt{25} = 1.2$; $z = (52.4 - 50) \div 1.2 = 2.0000$. Since 2.0000 exceeds 1.6449, reject H_0 . (b) Reject when $\bar{x}$ exceeds $50 + 1.6449(1.2) = 51.9738$ ml. (c) $\beta = P(\bar{x} \text{ less than } 51.9738 \mid \mu = 53) = \Phi((51.9738 - 53) \div 1.2) = \Phi(-0.8551) = 0.1962$, so the power is 0.8038. [3, 3, 3]

9. The number of accidents at a junction in a month has a Poisson distribution with mean 50. (a) Use a normal approximation with a continuity correction to estimate the probability of at most 42 accidents in a month. (b) Estimate the probability of between 45 and 55 accidents inclusive. (c) State one condition for the approximation to be reliable and comment on whether it is met here.

答案： $X \sim Po(50)$ has mean 50 and sd $\sqrt{50} = 7.0711$. (a) $P(X \leq 42) \approx \Phi((42.5 - 50) \div 7.0711) = \Phi(-1.0607) = 0.1444$. (b) $P(45 \leq X \leq 55) \approx \Phi(0.7778) - \Phi(-0.7778) = 0.5633$. (c) The approximation needs λ to be large (in practice at least about 10 - 15); here $\lambda = 50$ and the distribution is only slightly skew, so the approximation is acceptable although the continuity correction is essential. [3, 3, 3]

