Cambridge International AS & A Level — Further Mathematics (9231)

Paper 1 Further Pure Mathematics 1 — Answers and Mark Scheme

Time allowed: 2 hours	Total: 75 marks	8 questions
Answer **all** questions.	Practice paper — not a Cambridge paper	Prepared by TutorZone

Show all necessary working; answers without working may not receive full marks.

1. The cubic equation $x^3 - 6x^2 + 11x - 6 = 0$ has roots α, β and γ. **[8]**

(a) Find $\Sigma\alpha^2$ and $\Sigma\alpha^2\beta^2$. (4 marks)

(b) Find the cubic equation, with integer coefficients, whose roots are $\alpha + 1$, $\beta + 1$ and $\gamma + 1$. (4 marks)

Answer 1 (8 marks)

(a) By Vieta: $\Sigma\alpha = 6$, $\Sigma\alpha\beta = 11$, $\alpha\beta\gamma = 6$.

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = 36 - 22 = 14$.

$\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\Sigma\alpha = 121 - 2(6)(6) = 121 - 72 = 49$.

(b) Sum of new roots $= \Sigma\alpha + 3 = 9$; sum of products taken two at a time $= \Sigma\alpha\beta + 2\Sigma\alpha + 3 = 11 + 12 + 3 = 26$; product $= \alpha\beta\gamma + \Sigma\alpha\beta + \Sigma\alpha + 1 = 6 + 11 + 6 + 1 = 24$.

Hence $x^3 - 9x^2 + 26x - 24 = 0$ (roots 2, 3 and 4).

Marking notes: (a) 2 marks per expression; (b) 1 mark per symmetric sum and 1 mark for the equation.

2. The matrix $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. **[9]**

(a) Find the two eigenvalues of A and an eigenvector for each. (6 marks)

(b) Find A^{-1} and hence solve $4x + y = 9$, $2x + 3y = 7$. (3 marks)

Answer 2 (9 marks)

(a) $\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$, so $\lambda = 5$ or $\lambda = 2$.

For $\lambda = 5$: $(4 - 5)x + y = 0$, giving the eigenvector $(1, 1)$ (check $A(1, 1) = (5, 5) = 5(1, 1)$).

For $\lambda = 2$: $(4 - 2)x + y = 0$, giving the eigenvector $(1, -2)$ (check $A(1, -2) = (2, -4) = 2(1, -2)$).

(b) $\det A = 12 - 2 = 10$, so $A^{-1} = (1/10)\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$.

$x = (3 \times 9 - 1 \times 7) / 10 = 2$ and $y = (-2 \times 9 + 4 \times 7) / 10 = 1$.

Hence $x = 2$, $y = 1$.

Marking notes: 2 marks for the characteristic equation, 1 mark per eigenvalue, 1 mark per eigenvector; 2 marks for A^{-1} and 1 mark for the solution.

3. (a) Using the standard summation formulae, express $\Sigma(r = 1 \text{ to } n)\ r(2r + 3)$ in terms of n. (4 marks) **[8]**

(b) Using partial fractions and the method of differences, prove that $\Sigma(r = 1 \text{ to } n)\ 1/((2r - 1)(2r + 1)) = n/(2n + 1)$. (4 marks)

Answer 3 (8 marks)

(a) $\Sigma r(2r + 3) = 2\Sigma r^2 + 3\Sigma r = 2 \times n(n + 1)(2n + 1)/6 + 3 \times n(n + 1)/2$.

Combining over a common denominator gives $n(n + 1)[2(2n + 1) + 9]/6 = n(n + 1)(4n + 11)/6$.

Check $n = 4$: $4 \times 5 \times 27/6 = 90$, and $5 + 14 + 27 + 44 = 90$.

(b) $1/((2r - 1)(2r + 1)) = (1/2)[1/(2r - 1) - 1/(2r + 1)]$.

Summing cancels all the intermediate terms: $(1/2)[1 - 1/(2n + 1)] = n/(2n + 1)$.

Marking notes: (a) 1 mark per standard formula and 2 marks for the algebra; (b) 2 marks for partial fractions, 1 for the cancelling and 1 for the conclusion.

4. Let $z = 2 + 2\sqrt{3}\,i$. **[10]**

(a) Find $|z|$ and $\arg z$, giving the argument as a multiple of π. (3 marks)

(b) Find z^3 in the form $a + bi$. (3 marks)

(c) Hence solve $w^3 = -64$, giving the three roots in the form $r(\cos\theta + i\sin\theta)$. (4 marks)

Answer 4 (10 marks)

(a) $|z| = \sqrt{(4 + 12)} = 4$. $\tan\theta = 2\sqrt{3} / 2 = \sqrt{3}$ and z is in the first quadrant, so $\arg z = \pi/3$, i.e. $z = 4(\cos\pi/3 + i\sin\pi/3)$.

(b) $z^3 = 4^3(\cos\pi + i\sin\pi) = 64(-1 + 0i) = -64$.

(c) From (b), $w = 4(\cos\pi/3 + i\sin\pi/3)$ is one root; the others have arguments increased by $2\pi/3$ and $4\pi/3$: $4(\cos\pi/3 + i\sin\pi/3) = 2 + 2\sqrt{3}\,i$; $4(\cos\pi + i\sin\pi) = -4$; $4(\cos 5\pi/3 + i\sin 5\pi/3) = 2 - 2\sqrt{3}\,i$.

Marking notes: (a) 2 marks for the modulus, 1 for the argument; (b) 3 marks; (c) 1 mark per root and 1 mark for using multiples of π.

5. The plane Π has equation $2x - y + 2z = 9$. The point A is $(1, -1, 3)$ and the line l has equation $r = (3, 1, 2) + t(1, -2, 2)$. **[10]**

(a) Show that A lies on Π and find the point where l meets Π. (4 marks)

(b) Find the perpendicular distance from B$(5, 1, 4)$ to Π. (3 marks)

(c) Find the angle θ between l and Π, giving $\sin\theta$. (3 marks)

Answer 5 (10 marks)

(a) At A: $2(1) - (-1) + 2(3) = 2 + 1 + 6 = 9$, so A lies on Π.

Substituting l into Π: $2(3 + t) - (1 - 2t) + 2(2 + 2t) = 9 + 8t$. Setting $9 + 8t = 9$ gives $t = 0$, so the intersection is $(3, 1, 2)$.

(b) At B: $2(5) - 1 + 2(4) = 17$. The normal has length $\sqrt{(4 + 1 + 4)} = 3$, so the distance is $|17 - 9| / 3 = 8/3$.

(c) With $d = (1, -2, 2)$ and $n = (2, -1, 2)$: $d \cdot n = 2 + 2 + 4 = 8$ and $|d| = |n| = 3$, so $\sin\theta = 8/9$ ($\theta \approx 62.7°$).

Marking notes: (a) 2 marks for verifying A and 2 for the intersection; (b) 2 marks for the substitution and normal length, 1 for the answer; (c) 1 mark for the dot product, 1 for the moduli, 1 for the answer.

6. Prove by induction that, for all positive integers n, $\Sigma(r = 1 \text{ to } n)\ r\,2^r = (n - 1)2^{(n+1)} + 2$. **[10]**

Answer 6 (10 marks)

For $n = 1$: LHS $= 1 \times 2 = 2$ and RHS $= 0 \times 4 + 2 = 2$, so the statement holds.

Assume it holds for some positive integer k, i.e. $\Sigma(r = 1 \text{ to } k)\ r\,2^r = (k - 1)2^{(k+1)} + 2$.

Then $\Sigma(r = 1 \text{ to } k+1)\ r\,2^r = (k - 1)2^{(k+1)} + 2 + (k + 1)2^{(k+1)}$

$= 2^{(k+1)}[(k - 1) + (k + 1)] + 2 = 2k \cdot 2^{(k+1)} + 2 = k\,2^{(k+2)} + 2$

$= ((k + 1) - 1)2^{((k+1)+1)} + 2$, so the statement holds for $n = k + 1$.

By mathematical induction the statement holds for all positive integers n.

Marking notes: 2 marks for the base case, 2 for the inductive assumption, 4 for the algebra and 2 for the concluding statement.

7. Let $f(x) = \sqrt{(1 + 2x)}$. **[10]**

(a) Find $f'(x)$, $f''(x)$ and $f'''(x)$. (4 marks)

(b) Hence find the Maclaurin expansion of $f(x)$, up to and including the term in x^3. (3 marks)

(c) Use the expansion in (b) to estimate $\sqrt{1.2}$ and state the difference from the true value in scientific notation. (3 marks)

Answer 7 (10 marks)

(a) $f'(x) = (1 + 2x)^{(-1/2)}$, $f''(x) = -(1 + 2x)^{(-3/2)}$ and $f'''(x) = 3(1 + 2x)^{(-5/2)}$.

Hence $f(0) = 1$, $f'(0) = 1$, $f''(0) = -1$ and $f'''(0) = 3$.

(b) The Maclaurin expansion is $1 + x + (-1)x^2/2! + 3x^3/3! = 1 + x - x^2/2 + x^3/2$.

(c) With $x = 0.1$: $1 + 0.1 - 0.005 + 0.0005 = 1.0955$.

The true value is $\sqrt{1.2} = 1.095445\ldots$, so the difference is 5.5×10^{-5} (about 5.49×10^{-5}).

Marking notes: (a) 1 mark per derivative and 1 for the four values; (b) 3 marks for the expansion; (c) 1 mark each for the substitution, the true value and the difference.

8. The polar curve C has equation $r = a(1 + \cos\theta)$, $0 \le \theta \le 2\pi$, where $a > 0$. **[10]**

(a) Show that the area enclosed by C is $(3/2)\pi a^2$. (6 marks)

(b) Taking $a = 2$, find this area (exact value and to 3 significant figures) and the value of r at $\theta = \pi/3$. (4 marks)

Answer 8 (10 marks)

(a) Area $= (1/2)\int_{0}^{2\pi} r^2\, d\theta = (a^2/2)\int_{0}^{2\pi} (1 + \cos\theta)^2\, d\theta$.

Expanding gives $1 + 2\cos\theta + \cos^2\theta$, with $\cos^2\theta = (1 + \cos 2\theta)/2$.

$\int_{0}^{2\pi} 1\, d\theta = 2\pi$, $\int_{0}^{2\pi} 2\cos\theta\, d\theta = 0$ and $\int_{0}^{2\pi} \cos^2\theta\, d\theta = \pi$.

The total is 3π, so the area is $(a^2/2)(3\pi) = (3/2)\pi a^2$.

(b) With $a = 2$: area $= (3/2)\pi(4) = 6\pi \approx 18.8$.

At $\theta = \pi/3$, $\cos\theta = 1/2$, so $r = 2(1 + 1/2) = 3$.

Marking notes: (a) 2 marks for the area formula, 2 for the expansion and half-angle, 1 for integrating term by term and 1 for the conclusion; (b) 2 marks for the exact value, 1 for the decimal and 1 for $r = 3$.

Original questions written by TutorZone for practice. This paper is not a Cambridge past paper, specimen paper or mark scheme. Paper format follows the published 9231 assessment structure (2 hours, 75 marks, 6–8 structured questions).