

HKDSE Mathematics Extended Part Module 2 (Algebra and Calculus) Mock Paper (10 questions, 100 marks, 2 hours 30 minutes)

中六 (HKDSE 選修單元) Mathematics | 時限 : 150 分鐘 | 總分 : 100 分 | 共 10 題

Section A (Questions 1 to 6, 50 marks)

Answer ALL questions. Show your working, give numerical answers to three significant figures unless told otherwise, and state units in every final answer. All data are examples and this is not a past paper of any examination board.

1. Let $S(n)$ be the statement "for every positive integer n , the sum of $r \cdot 2^r$ from $r = 1$ to n equals $(n - 1) \cdot 2^{(n+1)} + 2$ ".
(a) Prove $S(n)$ by mathematical induction for all positive integers n .
(b) Hence find the sum of $r \cdot 2^r$ from $r = 1$ to 5.

答案 : (a) When $n = 1$: LHS = $1 \times 2 = 2$ and RHS = $(1 - 1) \times 4 + 2 = 2$, so the statement holds.
Assume it holds for some positive integer k , that is the sum from $r = 1$ to k equals $(k - 1) \cdot 2^{(k+1)} + 2$.
For $n = k + 1$ the sum is $(k - 1) \cdot 2^{(k+1)} + 2 + (k + 1) \cdot 2^{(k+1)} = (2k) \cdot 2^{(k+1)} + 2 = k \cdot 2^{(k+2)} + 2 = [(k + 1) - 1] \cdot 2^{(k+1)+1} + 2$, so the statement also holds for $n = k + 1$.
By the principle of mathematical induction, $S(n)$ is true for all positive integers n .
(b) Put $n = 5$: the sum is $4 \times 64 + 2 = 258$.
Marking: (a) 6 marks (base case 1, induction assumption 1, substitution 2, simplification 2); (b) 2 marks.

2. In the expansion of $(2 - x/2)^8$,
(a) find the constant term;
(b) find the coefficient of x^3 ;
(c) find the coefficient of x^4 .

答案 : (a) Constant term = $C(8,0) \times 2^8 = 256$.
(b) The term in x^3 is $C(8,3) \times 2^5 \times (-1/2)^3 = 56 \times 32 \times (-1/8) = -224$, so the coefficient is -224.
(c) The term in x^4 is $C(8,4) \times 2^4 \times (-1/2)^4 = 70 \times 16 \times (1/16) = 70$, so the coefficient is 70.
Marking: (a) 2 marks; (b) 4 marks (binomial coefficient, power of 2, sign, final answer); (c) 2 marks.

3. (a) Find the limit of $(x^2 - 5x + 6)/(x - 2)$ as x tends to 2.
(b) Let $h(x) = x^2 \ln x$ for x greater than 0. Find $h'(1)$.
(c) Find the equation of the tangent to $y = h(x)$ at $x = 1$.

答案 : (a) Since $x^2 - 5x + 6 = (x - 2)(x - 3)$, the quotient is $x - 3$ for x not equal to 2, so the limit is $2 - 3 = -1$.
(b) $h'(x) = 2x \ln x + x^2 \times (1/x) = 2x \ln x + x$, hence $h'(1) = 0 + 1 = 1$.
(c) $h(1) = 0$, so the tangent is $y = 1 \times (x - 1)$, that is $y = x - 1$.
Marking: (a) 3 marks (factorisation 1, cancellation 1, answer 1); (b) 3 marks; (c) 2 marks.

4. (a) Use the substitution $u = x^2 + 3$ to find the integral of $x \cdot \sqrt{x^2 + 3}$ with respect to x .
(b) Hence evaluate the definite integral of $x \cdot \sqrt{x^2 + 3}$ from $x = 1$ to $x = 3$, giving the answer to three significant figures.

答案 : (a) Let $u = x^2 + 3$, then $du = 2x \, dx$, so $x \, dx = du/2$.
The integral becomes $(1/2) \cdot \int u^{(1/2)} \, du = (1/2) \cdot (2/3) u^{(3/2)} + C = (1/3)(x^2 + 3)^{(3/2)} + C$.
(b) Value = $(1/3)(12^{(3/2)} - 4^{(3/2)}) = (1/3)(41.569 - 8) = 11.190$, that is about 11.2.
Marking: (a) 5 marks (substitution 2, antiderivative 2, constant 1); (b) 4 marks (limits 2, value 2).

5. (a) Solve $2\sin^2(\theta) - \sin(\theta) - 1 = 0$ for $0 \leq \theta \leq 360$ degrees. (b) Hence solve $2\sin^2(2x) - \sin(2x) - 1 = 0$ for $0 \leq x \leq 180$ degrees.

答案 : (a) $2\sin^2(\theta) - \sin(\theta) - 1 = (2\sin(\theta) + 1)(\sin(\theta) - 1) = 0$. If $\sin(\theta) = 1$ then $\theta = 90$ degrees; if $\sin(\theta) = -0.5$ then $\theta = 210$ or 330 degrees. So $\theta = 90, 210$ or 330 degrees. (b) Taking $2x$ equal to each value in (a) with $0 \leq 2x \leq 360$ degrees gives $2x = 90, 210$ or 330 degrees, so $x = 45, 105$ or 165 degrees. Marking: (a) 5 marks (factorisation 2, three roots 3); (b) 4 marks (range 1, three roots 3).

6. Let A be the matrix with rows $(2, 1)$ and $(3, 4)$. (a) Find $\det A$ and A^{-1} . (b) Using the inverse matrix, solve $2x + y = 5$ and $3x + 4y = 6$.

答案 : (a) $\det A = 2 \times 4 - 1 \times 3 = 5$; $A^{-1} = (1/5) \times$ rows $(4, -1)$ and $(-3, 2)$, that is rows $(0.8, -0.2)$ and $(-0.6, 0.4)$. (b) $(x; y) = A^{-1} (5; 6) = (1/5) \times (14; -3) = (2.8; -0.6)$, so $x = 2.8$ and $y = -0.6$. Marking: (a) 4 marks (determinant 1, cofactors 2, inverse 1); (b) 4 marks (matrix product 2, two values 2).

Section B (Questions 7 to 10, 50 marks)

Answer ALL questions. Show your working, give numerical answers to three significant figures unless told otherwise, and state units in every final answer. All data are examples and this is not a past paper of any examination board.

7. Consider the system $x + y + z = 6$, $2x - y + z = 3$ and $x + 2y - kz = 3$, where k is a real number. (a) Express the determinant of the coefficient matrix in terms of k . (b) Find the value of k for which the system does not have a unique solution. (c) When $k = 1$, solve the system by Cramer's rule.

答案 : (a) Determinant $= 1[(k) - 2] - 1[-2k - 1] + 1[4 + 1] = 3k + 4$. (b) No unique solution means the determinant is zero, so $3k + 4 = 0$ and $k = -4/3$. (c) When $k = 1$ the determinant is 7; by Cramer's rule $x = 9/7$, $y = 15/7$ and $z = 18/7$, which satisfy all three equations on substitution. Marking: (a) 4 marks; (b) 3 marks; (c) 5 marks (three determinants 3, three values 2).

8. Let $a = i + 2j - 2k$ and $b = 2i - j + k$. (a) Find $|a|$, $|b|$ and the scalar product $a \cdot b$. (b) Find the angle between a and b , giving the answer to three significant figures. (c) Find the vector product $a \times b$ and $|a \times b|$. (d) Find the scalar component of b along a .

答案 : (a) $|a| = 3$, $|b| = \sqrt{6} = 2.45$ and $a \cdot b = 2 - 2 - 2 = -2$. (b) $\cos(\theta) = -2 / (3\sqrt{6}) = -0.2722$, so $\theta = 106$ degrees (three significant figures). (c) $a \times b = 0i - 5j - 5k$ and $|a \times b| = \sqrt{50} = 7.07$. (d) Scalar component $= a \cdot b / |a| = -2/3 = -0.667$; the negative sign shows the opposite direction to a . Marking: (a) 3 marks; (b) 3 marks (cosine 2, angle 1); (c) 4 marks (cross product 3, magnitude 1); (d) 2 marks.

9. The curve C has equation $y = x^3 - 6x^2 + 9x + 1$. (a) Find the stationary points of C and determine their nature. (b) Find the point of inflexion of C . (c) Find the area of the region bounded by C , the lines $x = 0$, $x = 3$ and the x -axis.

答案 : (a) $y' = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so stationary points at $x = 1$ and $x = 3$. Since $y'' = 6x - 12$ is negative at $x = 1$ and positive at $x = 3$, $(1, 5)$ is a maximum point and $(3, 1)$ is a minimum point. (b) $y'' = 0$ at $x = 2$ and changes sign there, so $(2, 3)$ is a point of inflexion. (c) Area $= 81/4 - 54 + 81/2 + 3 = 39/4 = 9.75$ square units. Marking: (a) 6 marks (derivative 1, stationary points 2, second derivative test 3); (b) 3 marks; (c) 4 marks (antiderivative 2, substitution 2).

10. (a) Use integration by parts to find the integral of $x^2 e^{(2x)}$ with respect to x .
(b) Hence evaluate the definite integral of $x^2 e^{(2x)}$ from $x = 0$ to $x = 1$, giving the answer to three significant figures.
(c) Explain why the substitution $u = e^{(2x)}$ cannot be used directly for part (a).

答案：(a) Take $u = x^2$ and $dv = e^{(2x)} dx$, so $du = 2x dx$ and $v = e^{(2x)}/2$. Then the integral equals $(x^2 e^{(2x)})/2 - \int x e^{(2x)} dx$. Applying parts again gives $e^{(2x)}(x^2/2 - x/2 + 1/4) + C$.
(b) Substituting the limits gives $0.25 e^2 - 0.25 = 1.597$, that is about 1.60.
(c) With $u = e^{(2x)}$ we get $du = 2 e^{(2x)} dx$, and the factor x^2 cannot be written in terms of u alone without logarithms, which makes the integral harder, so the substitution is not useful.
Marking: (a) 7 marks (first application 3, second application 3, constant 1); (b) 4 marks; (c) 2 marks.