S6 Mathematics (Extended Part Module 1) Practice Paper: Binomial Theorem and Sequences, Exponentials and Logarithms, Differentiation, Integration and Applications, Binomial and Poisson Distributions, Normal Distribution and Point Estimation

Level: S6 | Subject: Mathematics (Module 1) | 75 minutes | 100 marks | 24 questions | answer key

Section A: Binomial Theorem and Sequences (18 marks)

Show all working. Binomial coefficients may be written as C(n, r); state the formula before substituting.

#	Question	Answer	Marks
1	Expand (1 + 2x)^5, and state the coefficients of x^3 and x^2.	By the binomial theorem the coefficient of x^3 is C(5,3) (2^3) = 10 x 8 = 80, and the coefficient of x^2 is C(5,2) (2^2) = 10 x 4 = 40. The expansion is 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5 (check with x = 1: sum = 243 = 3^5).	5
2	In the expansion of (2 + x)^6, find the constant term and the coefficient of x^4.	The general term is C(6,r) 2^(6-r) x^r. Constant term (r = 0) = 2^6 = 64; coefficient of x^4 = C(6,4)(2^2) = 15 x 4 = 60.	5
3	An arithmetic sequence has first term 5 and common difference 3. Find the 10th term and the sum of the first 10 terms.	T(10) = a + 9d = 5 + 9 x 3 = 32; S(10) = (10/2)[2a + 9d] = 5 x (10 + 27) = 185.	4
4	A geometric sequence has first term 3 and common ratio 2. Find the 6th term and the sum of the first 6 terms.	T(6) = a r^5 = 3 x 32 = 96; S(6) = a(r^6 - 1)/(r - 1) = 3(64 - 1)/1 = 189.	4

Section B: Exponentials and Logarithms (17 marks)

State the log laws used. Give non-integer answers to 3 significant figures.

#	Question	Answer	Marks
5	Solve 2^(x + 1) = 32.	Since 32 = 2^5, x + 1 = 5 and x = 4.	4
6	Without a calculator, evaluate (a) log base 3 of 81, (b) log base 2 of 1/8.	(a) 81 = 3^4 so log_3 81 = 4; (b) 1/8 = 2^-3 so log_2 (1/8) = -3.	4
7	Solve log_2 x + log_2 (x - 2) = 3 and explain why one root is rejected.	Combining gives log_2[x(x - 2)] = 3, so x(x - 2) = 8, x^2 - 2x - 8 = 0, (x - 4)(x + 2) = 0. Thus x = 4 or x = -2; since the argument of a log must be positive we need x > 2, so only x = 4 is accepted.	5
8	A sum of $5000 is deposited at 5% per annum compounded yearly. After how many complete years will the amount first be at least $8000?	Solve 5000(1.05)^n >= 8000, i.e. 1.05^n >= 1.6, giving n >= log 1.6 / log 1.05 = 9.633, so n = 10 years (amount = $8,144.47).	4

Section C: Differentiation (17 marks)

State the rule used (chain, product or implicit). Simplify your answers.

#	Question	Answer	Marks
9	Let y = (2x + 3)^5. Find dy/dx and its value when x = 1.	By the chain rule dy/dx = 5(2x + 3)^4 x 2 = 10(2x + 3)^4; at x = 1 this is 10 x 5^4 = 6250.	5
10	Let y = x^2 ln x (x > 0). Find dy/dx and its value at x = e (3 significant figures).	By the product rule dy/dx = 2x ln x + x^2(1/x) = 2x ln x + x; at x = e this is 2e(1) + e = 3e = 8.15.	4
11	The curve x^2 + y^2 = 25 passes through (3, 4). Find the slope of the tangent there.	Differentiating implicitly: 2x + 2y(dy/dx) = 0, so dy/dx = -x/y; at (3, 4) the slope is -3/4 = -0.75.	4
12	Evaluate lim(x->0) (sin 3x)/x.	Write it as 3 x (sin 3x)/(3x); since (sin t)/t -> 1 as t -> 0, the limit is 3.	4

Section D: Integration and Applications (16 marks)

Include the constant C for indefinite integrals; write the primitive before substituting limits.

#	Question	Answer	Marks
13	Evaluate the definite integral of 3x^2 from 0 to 2.	The primitive is x^3, so the integral equals [x^3] from 0 to 2 = 8 - 0 = 8.	4
14	Find the indefinite integral of (2x + 1)^3 with respect to x.	Since d/dx (2x + 1)^4 = 8(2x + 1)^3, the integral is (2x + 1)^4/8 + C.	4
15	Find the area enclosed between y = x and y = x^2 for 0 <= x <= 1.	On 0 < x < 1 we have x > x^2, so the area is the integral from 0 to 1 of (x - x^2) = [x^2/2 - x^3/3] = 1/2 - 1/3 = 1/6 = 0.1667.	4
16	Evaluate the integral of 1/x from 1 to e and explain the result.	The integral equals [ln x] from 1 to e = ln e - ln 1 = 1.0; that is, the area under y = 1/x between x = 1 and x = e is 1 square unit.	4

Section E: Binomial and Poisson Distributions (16 marks)

State the distribution and its parameters; give probabilities to 4 decimal places.

#	Question	Answer	Marks
17	Let X ~ B(10, 0.3). Find (a) P(X = 3), (b) E(X) and Var(X).	(a) P(X = 3) = C(10,3)(0.3)^3(0.7)^7 = 0.2668; (b) E(X) = np = 3.0 and Var(X) = np(1 - p) = 2.0999999999999996.	4
18	Let X ~ Po(4). Find P(X = 2) to 4 decimal places.	P(X = 2) = e^-4 x 4^2 / 2! = 8e^-4 = 0.1465.	4
19	A switchboard receives on average 2 calls per minute, modelled by a Poisson distribution. Find the probability of no calls in a given minute.	X ~ Po(2), so P(X = 0) = e^-2 = 0.1353.	4
20	A shooter hits the target with probability 0.4 on each independent shot, and fires 8 shots. Find the probability of at least one hit (4 decimal places).	X ~ B(8, 0.4) and P(X >= 1) = 1 - P(X = 0) = 1 - 0.6^8 = 1 - 0.0168 = 0.9832.	4

Section F: Normal Distribution and Point Estimation (16 marks)

The standard normal table may be used; give table values to 4 decimal places and describe the shaded region.

#	Question	Answer	Marks
21	Let Z ~ N(0, 1). Find (a) P(Z < 1.96) and (b) P(-1 < Z < 1).	(a) From the table P(Z < 1.96) = 0.975; (b) P(-1 < Z < 1) = 0.8413 - 0.1587 = 0.6826000000000001.	4
22	Let X ~ N(50, 100). Find P(X > 60).	Standardising, z = (60 - 50)/10 = 1, so P(X > 60) = P(Z > 1) = 1 - 0.8413 = 0.1587.	4
23	For the sample 12, 15, 18, 20, 25, find the sample mean and sample standard deviation (3 significant figures).	Mean = 18.0; sum of squared deviations = 98.0; sample variance = 98.0/4 = 24.5; sample standard deviation = 4.950.	4
24	A sample of size 36 has mean 52 and standard deviation 6. Construct a 95% confidence interval for the population mean.	Standard error = 6/6 = 1.0; the 95% interval is 52 +/- 1.96 x 1.0 = (50.04, 53.96).	4