S6 Mathematics Extended Part Module 2 Practice Paper

S6 | Mathematics (Module 2) | Time: 75 minutes | Total: 100 marks | 24 questions

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Part A: Mathematical Induction and Binomial Theorem (12 marks)

Show all working; give coefficients in simplest form.

1. Find the coefficient of x^3 in the expansion of (1 + 2x)^5.

Answer: The general term is 5Cr (2x)^r; take r = 3: 5C3 × 2^3 = 10 × 8 = 80.

2. Find the constant term in the expansion of (x - 3/x)^6.

Answer: The general term is 6Cr x^(6-r) (-3/x)^r; the constant term needs 6-2r = 0, so r = 3: 6C3 × (-3)^3 = 20 × (-27) = -540.

3. Use mathematical induction to prove that 1 + 3 + 5 + … + (2n - 1) = n^2 for all positive integers n.

Answer: For n = 1, LHS = 1 = 1^2, true. Assume true for n = k, i.e. the sum of the first k terms is k^2; then for n = k+1 the sum is k^2 + (2k+1) = (k+1)^2, so it holds; by induction the statement is true. Check n = 4: 1+3+5+7 = 16 = 4^2.

4. Find the coefficient of x^2 in the expansion of (2 + x)^7.

Answer: Take r = 2: 7C2 × 2^5 = 21 × 32 = 672.

Part B: Trigonometric Identities and Equations (16 marks)

Unless otherwise stated, give answers to 3 significant figures or in surd form.

5. Given that A and B are acute angles with sin A = 3/5 and cos B = 5/13, find the exact values of sin(A + B) and cos(A + B).

Answer: cos A = 4/5 and sin B = 12/13. sin(A+B) = sinA cosB + cosA sinB = 36/65 + 20/65 = 56/65; cos(A+B) = cosA cosB - sinA sinB = 48/65 - 15/65 = 33/65.

6. Solve 2sin^2 x - sin x - 1 = 0 for 0° ≤ x ≤ 360°.

Answer: Factorising gives (2sin x + 1)(sin x - 1) = 0, so sin x = 1 or sin x = -1/2. sin x = 1 gives x = 90°; sin x = -1/2 gives x = 210° or 330°. Answers: 90°, 210°, 330°.

7. Find the maximum value of 3sin x + 4cos x and the value of tan x at that maximum.

Answer: Write it as R sin(x + α) with R = √(3^2 + 4^2) = 5, so the maximum is 5; then sin(x+α) = 1, i.e. x = 90° - α where tan α = 4/3, hence tan x = 1/tan α = 3/4 = 0.75.

8. Given cos A = 0.6, find the value of cos 2A.

Answer: cos 2A = 2cos^2 A - 1 = 2(0.6)^2 - 1 = 0.72 - 1 = -0.28.

Part C: Limits and Applications of Differentiation (20 marks)

Show the steps of differentiation or limit evaluation.

9. Evaluate lim(x→0) sin 3x / (5x).

Answer: sin 3x / (5x) = (3/5)·(sin 3x / (3x)); as x→0, sin3x/(3x)→1, so the limit is 3/5 = 0.60.

10. Let y = x^2 e^(3x). Find dy/dx, and evaluate it at x = 1 (to 3 significant figures).

Answer: By the product rule dy/dx = 2x e^(3x) + 3x^2 e^(3x) = x e^(3x)(2 + 3x). At x = 1, dy/dx = 5e^3 ≈ 100.4.

11. Find the maximum and minimum points of the curve y = x^3 - 3x^2 + 2.

Answer: dy/dx = 3x^2 - 6x = 3x(x - 2), so stationary points at x = 0 and x = 2. At x = 0, y = 2 and the second derivative is -6 < 0, giving a maximum point (0, 2); at x = 2, y = 8 - 12 + 2 = -2 and the second derivative is 6 > 0, giving a minimum point (2, -2).

12. Find the equation of the tangent to the curve y = x ln x at x = e.

Answer: y(e) = e ln e = e, so the point is (e, e). dy/dx = ln x + 1, so at x = e the slope is 1 + 1 = 2; the tangent is y - e = 2(x - e), i.e. y = 2x - e.

Part D: Integration, Area and Volume (20 marks)

For definite integrals, show the antiderivative step.

13. Evaluate ∫(from 0 to 1)(3x^2 + 2x) dx.

Answer: The antiderivative is x^3 + x^2; substituting the limits gives 1 + 1 = 2.

14. Evaluate ∫(from 0 to 1) x e^x dx.

Answer: By integration by parts, ∫x e^x dx = e^x(x - 1) + C; at x = 0 this is -1 and at x = 1 it is 0, so the definite integral is 0 - (-1) = 1.

15. Find the area of the region bounded by the curves y = x^2 and y = 2x - x^2.

Answer: The curves meet where x^2 = 2x - x^2, i.e. 2x^2 - 2x = 0, so x = 0 or 1. Area = ∫(0 to 1)(2x - 2x^2) dx = [x^2 - (2/3)x^3] = 1 - 2/3 = 1/3 ≈ 0.3333.

16. The curve y = x^2 from x = 0 to x = 1 is rotated through one complete turn about the x-axis. Find the volume of the solid generated.

Answer: V = π∫(0 to 1)(x^2)^2 dx = π[x^5/5] (0 to 1) = π/5 ≈ 0.6283.

Part E: Vectors (16 marks)

Let a = (2, -1, 3) and b = (1, 4, -2).

17. Find a · b.

Answer: a · b = (2)(1) + (-1)(4) + (3)(-2) = 2 - 4 - 6 = -8.

18. Find the angle between a and b (to 3 significant figures).

Answer: |a| = √14, |b| = √21, cos θ = -8/√(294) ≈ -0.4665, so θ ≈ 117.8°.

19. Find a × b and its magnitude (magnitude to 3 significant figures).

Answer: a × b = ((-1)(-2) - (3)(4), (3)(1) - (2)(-2), (2)(4) - (-1)(1)) = (-10, 7, 9); magnitude = √(100 + 49 + 81) = √230 ≈ 15.17.

20. Find a + 2b and the scalar product of this vector with b.

Answer: a + 2b = (2 + 2, -1 + 8, 3 - 4) = (4, 7, -1); the scalar product with b is (4)(1) + (7)(4) + (-1)(-2) = 4 + 28 + 2 = 34, which also equals a·b + 2|b|^2 = -8 + 42 = 34.

Part F: Matrices and Linear Transformations (16 marks)

Let A = [[2, 1], [3, 4]] and B = [[1, 2], [0, 3]].

21. Find det A and the inverse of A.

Answer: det A = (2)(4) - (1)(3) = 5; A^-1 = (1/5)[[4, -1], [-3, 2]].

22. Find AB and hence det(AB).

Answer: AB = [[2(1)+1(0), 2(2)+1(3)], [3(1)+4(0), 3(2)+4(3)]] = [[2, 7], [3, 18]]; det(AB) = 36 - 21 = 15 = det A × det B = 5 × 3.

23. Use the inverse matrix to solve the system 2x + y = 8, 3x + 4y = 27.

Answer: x = (4×8 - 1×27)/5 = 5/5 = 1 and y = (-3×8 + 2×27)/5 = 30/5 = 6. Check: 2(1) + 6 = 8 and 3(1) + 4(6) = 27.

24. A linear transformation T is represented by the matrix M = [[0, -1], [1, 0]]. Describe the geometric effect of T and find its area scale factor.

Answer: M maps (x, y) to (-y, x), i.e. an anticlockwise rotation of 90° about the origin; the area scale factor is |det M| = |(0)(0) - (-1)(1)| = 1, so areas are unchanged.
