S4 Mathematics Extended Part Module 2 Practice Paper (Set 1) | Mathematical induction, binomial theorem, trigonometry, limits and differentiation, integration, matrices and vectors

S4 Mathematics Extended Part Module 2 | Time allowed: 75 minutes | Total: 100 marks | 24 questions

Section A: Mathematical induction and the binomial theorem

Question 1 (16 marks)

1. Prove by mathematical induction that for all positive integers n, 1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6. (4 marks)

> **Answer:** For n = 1: LHS = 1 and RHS = 1 x 2 x 3 / 6 = 1, so the statement holds. Assume it holds for some positive integer k, i.e. 1^2 + ... + k^2 = k(k+1)(2k+1)/6. Then 1^2 + ... + k^2 + (k+1)^2 = k(k+1)(2k+1)/6 + (k+1)^2 = (k+1)[k(2k+1) + 6(k+1)]/6 = (k+1)(2k^2 + 7k + 6)/6 = (k+1)(k+2)(2k+3)/6, so the statement also holds for k+1. By induction it holds for all positive integers n.

2. Using the formula in the previous part, evaluate 10^2 + 11^2 + ... + 15^2. (4 marks)

> **Answer:** 10^2 + ... + 15^2 = (1^2 + ... + 15^2) - (1^2 + ... + 9^2) = 15 x 16 x 31 / 6 - 9 x 10 x 19 / 6 = 1240 - 285 = 955.

3. Find the constant term in the expansion of (2x - 3/x)^6. (4 marks)

> **Answer:** The general term is C(6,r)(2x)^(6-r)(-3/x)^r = C(6,r) 2^(6-r) (-3)^r x^(6-2r). For the constant term, 6 - 2r = 0, so r = 3 and the constant term is C(6,3) x 2^3 x (-3)^3 = 20 x 8 x (-27) = -4320.

4. Find the coefficient of x^4 in the expansion of (2x - 3/x)^6. (4 marks)

> **Answer:** Setting 6 - 2r = 4 gives r = 1, so the coefficient of x^4 is C(6,1) x 2^5 x (-3)^1 = 6 x 32 x (-3) = -576.

Section B: Trigonometry

Question 2 (16 marks)

5. Without using a calculator, find the exact value of sin 15 degrees. (4 marks)

> **Answer:** sin 15 deg = sin(45 - 30) = sin45 cos30 - cos45 sin30 = (sqrt2/2)(sqrt3/2) - (sqrt2/2)(1/2) = (sqrt6 - sqrt2)/4, about 0.2588.

6. Given that A and B are acute angles with sin A = 3/5 and cos B = 5/13, find sin(A + B). (4 marks)

> **Answer:** By Pythagoras, cos A = 4/5 and sin B = 12/13. sin(A+B) = sinA cosB + cosA sinB = (3/5)(5/13) + (4/5)(12/13) = 15/65 + 48/65 = 63/65 ≈ 0.9692.

7. Solve 2sin^2 x + sin x - 1 = 0 for 0 deg <= x <= 360 deg. (4 marks)

> **Answer:** Factorising: (2 sin x - 1)(sin x + 1) = 0, so sin x = 1/2 or sin x = -1. From sin x = 1/2, x = 30 deg or 150 deg; from sin x = -1, x = 270 deg. Hence x = 30, 150 or 270 degrees.

8. Prove the identity (1 - cos 2theta) / sin 2theta = tan theta. (4 marks)

> **Answer:** LHS = (1 - (1 - 2 sin^2 theta)) / (2 sin theta cos theta) = 2 sin^2 theta / (2 sin theta cos theta) = sin theta / cos theta = tan theta, so the identity holds.

Section C: Limits and differentiation

Question 3 (16 marks)

9. Evaluate lim(x->3) (x^2 - 9)/(x - 3). (4 marks)

> **Answer:** (x^2 - 9)/(x - 3) = (x - 3)(x + 3)/(x - 3) = x + 3 for x not equal to 3, so the limit is 3 + 3 = 6.

10. Let y = (3x^2 + 1)^4. Find the value of dy/dx when x = 1. (4 marks)

> **Answer:** dy/dx = 4(3x^2 + 1)^3 x 6x. When x = 1, dy/dx = 4 x 4^3 x 6 = 1536.

11. Find the equation of the tangent to the curve y = x^3 - 3x at x = 2. (4 marks)

> **Answer:** dy/dx = 3x^2 - 3, so the slope at x = 2 is 9. The point on the curve is (2, 8 - 6) = (2, 2). The tangent is y - 2 = 9(x - 2), i.e. y = 9x - 16.

12. Evaluate lim(x->0) (1 - cos x)/x^2. (4 marks)

> **Answer:** Using 1 - cos x = 2 sin^2(x/2): (1 - cos x)/x^2 = 2 sin^2(x/2)/x^2. As x tends to 0, sin(x/2)/(x/2) tends to 1, so the limit is 2 x (1/2)^2 = 1/2.

Section D: Integration

Question 4 (16 marks)

13. Evaluate the definite integral of (2x + 1) from 0 to 3. (4 marks)

> **Answer:** The antiderivative is x^2 + x, so the integral from 0 to 3 is (9 + 3) - 0 = 12.

14. Find the area of the region bounded by the line y = 2x and the curve y = x^2. (4 marks)

> **Answer:** The curves meet where x^2 = 2x, i.e. x = 0 or 2. Area = integral from 0 to 2 of (2x - x^2) dx = [x^2 - x^3/3] from 0 to 2 = 4 - 8/3 = 4/3 ≈ 1.333.

15. Evaluate the integral of 2x/(x^2 + 1) from 0 to 1. (4 marks)

> **Answer:** Let u = x^2 + 1, du = 2x dx; the integral becomes the integral of du/u from 1 to 2 = [ln u] from 1 to 2 = ln 2, about 0.6931.

16. The curve y = x^2 (0 <= x <= 2) is rotated through one revolution about the x-axis. Find the volume of the solid formed. (4 marks)

> **Answer:** Volume = pi x integral from 0 to 2 of y^2 dx = pi x integral from 0 to 2 of x^4 dx = pi [x^5/5] from 0 to 2 = 32 pi / 5, about 20.11.

Section E: Matrices and linear transformations

Question 5 (16 marks)

17. Let A = [[3, 1], [2, 4]]. Find det A and the inverse of A. (4 marks)

> **Answer:** det A = 3 x 4 - 1 x 2 = 10, which is non-zero, so A is invertible and its inverse is (1/10)[[4, -1], [-2, 3]].

18. Solve the simultaneous equations 3x + y = 7 and 2x + 4y = 18. (4 marks)

> **Answer:** From the first equation y = 7 - 3x; substituting into the second gives 2x + 4(7 - 3x) = 18, so -10x = -10, hence x = 1 and y = 4.

19. Write down the matrix for a rotation of 90 degrees anticlockwise, and find the image of the point (3, -2) under this transformation. (4 marks)

> **Answer:** The matrix for a 90 degree anticlockwise rotation is [[0, -1], [1, 0]]; multiplying it by the column (3, -2) gives (2, 3), so the image is (2, 3).

20. Let B = [[2, -1], [5, 3]]. Find det B and det(AB), where A is the matrix above. (4 marks)

> **Answer:** det B = 2 x 3 - (-1) x 5 = 11, and since det(AB) = det A x det B = 10 x 11 = 110.

Section F: Vectors

Question 6 (20 marks)

21. Let a = (2, -1, 3) and b = (1, 4, -2). Find the scalar product a.b and the magnitude of a (correct to 3 decimal places). (5 marks)

> **Answer:** a.b = 2 x 1 + (-1) x 4 + 3 x (-2) = 2 - 4 - 6 = -8, and the magnitude of a is sqrt(4 + 1 + 9) = sqrt 14, about 3.7417.

22. Find the angle between a and b, correct to 0.1 degree. (5 marks)

> **Answer:** The magnitude of b is sqrt(1 + 16 + 4) = sqrt 21. cos theta = a.b / (|a||b|) = -8 / sqrt 294, about -0.4666, so theta is about 117.8°.

23. Find the vector product a x b, and hence the area of the parallelogram with adjacent sides a and b. (5 marks)

> **Answer:** a x b = the determinant with rows i j k; 2 -1 3; 1 4 -2, which equals (-10, 7, 9). The area is |a x b| = sqrt(100 + 49 + 81) = √230 ≈ 15.166.

24. Find the vector projection of b onto a. (5 marks)

> **Answer:** The projection is (a.b / |a|^2) a = (-8/14)(2, -1, 3) = (-8/7, 4/7, -12/7).