S4 Mathematics Extended Part Module 1 Practice Paper: Binomial Theorem, Exponential and Logarithmic Equations, Limits and Derivatives, Differentiation Rules with Tangents and Normals, Extrema and Applications, Definite Integrals and the Trapezoidal Rule (100 marks)

S4 | Mathematics Extended Part Module 1 | Time: 75 minutes | Total: 100 marks | 24 questions

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Part 1: Binomial Theorem (16 marks)

Show the expansion or binomial coefficient method.

1. Find the coefficient of x^3 in the expansion of (1 + 2x)^5. [4]

Answer: Coefficient = C(5,3) x 2^3 = 10 x 8 = 80.

2. Find the coefficient of x^2 in the expansion of (2 - x)^6. [4]

Answer: Coefficient = C(6,2) x 2^4 x (-1)^2 = 15 x 16 = 240.

3. Find the coefficient of x^2 in (1 + 3x)(1 + x)^7. [4]

Answer: Coefficient of x^2 = C(7,2) + 3 x C(7,1) = 21 + 21 = 42.

4. Let n be a positive integer. If the coefficient of x^2 in (1 + x)^n is 45, find n. [4]

Answer: n(n-1)/2 = 45, so n^2 - n - 90 = 0 and n = 10.

Part 2: Indices and Logarithmic Equations (16 marks)

Give exact answers; use logarithms where the answer is not an integer.

5. Solve 3^(2x-1) = 27. [4]

Answer: 27 = 3^3, so 2x - 1 = 3 and x = 2.

6. Solve log2 x + log2 (x - 2) = 3. [4]

Answer: log2[x(x-2)] = 3, so x(x-2) = 8, x^2 - 2x - 8 = 0, x = 4 or x = -2; reject -2 since x > 2.

7. Solve e^(2x) - 5e^x + 6 = 0, giving answers in logarithmic form. [4]

Answer: Let u = e^x: u^2 - 5u + 6 = 0, so u = 2 or 3, giving x = ln 2 or x = ln 3 (about 0.693 or 1.099).

8. Simplify log_a 8 + log_a 2 - log_a 4, where a > 0 and a is not 1. [4]

Answer: Expression = log_a (8 x 2 / 4) = log_a 4 = 2 log_a 2.

Part 3: Limits and Differentiation from First Principles (18 marks)

Do not substitute directly; show valid limiting arguments.

9. Find lim (x→3) (x^2 - 9)/(x - 3). [4]

Answer: Factorise: (x-3)(x+3); cancel x-3 to get x + 3, which tends to 3 + 3 = 6.

10. Find lim (x→0) (sqrt(1 + x) - 1)/x. [4]

Answer: Multiply by the conjugate (sqrt(1+x) + 1): the expression becomes 1/(sqrt(1+x) + 1), tending to 1/2 = 0.5.

11. Let f(x) = x^2 + 3x. Use first principles to find f'(x), then evaluate f'(2). [5]

Answer: (f(x+h) - f(x))/h = 2x + h + 3, so as h tends to 0, f'(x) = 2x + 3 and f'(2) = 4 + 3 = 7.

12. Find lim (x→∞) (3x^2 + 5x)/(2x^2 - 1). [5]

Answer: Divide numerator and denominator by x^2: (3 + 5/x)/(2 - 1/x^2), which tends to 3/2 = 1.5.

Part 4: Differentiation Rules, Tangents and Normals (18 marks)

Write the derivative first, then substitute.

13. Given y = x^3 - 4x, find dy/dx at x = 2. [5]

Answer: dy/dx = 3x^2 - 4, so at x = 2 the value is 12 - 4 = 8.

14. Given y = (2x + 1)^5, find dy/dx at x = 1. [5]

Answer: Chain rule: dy/dx = 5(2x+1)^4 x 2 = 10(2x+1)^4; at x = 1 this is 10 x 81 = 810.

15. Find the equation of the tangent to y = x^2 at (3, 9). [4]

Answer: dy/dx = 2x = 6; y - 9 = 6(x - 3), so y = 6x - 9, i.e. 6x - y - 9 = 0.

16. Find the equation of the normal to y = x^2 at (3, 9). [4]

Answer: Normal slope = -1/6; y - 9 = -(x - 3)/6, so x + 6y - 57 = 0.

Part 5: Extrema and Related Rates (18 marks)

Justify extrema with the first derivative.

17. Find the local maximum and local minimum of y = x^3 - 3x^2 + 4. [4]

Answer: dy/dx = 3x^2 - 6x = 3x(x - 2); at x = 0, y = 4 (local maximum); at x = 2, y = 0 (local minimum).

18. A rectangle has perimeter 40 cm. Find its maximum area. [4]

Answer: Let length be x and width 20 - x; A = x(20 - x). A' = 20 - 2x = 0, so x = 10 and A = 10 x 10 = 100 cm^2.

19. A sphere has volume V = (4/3)πr^3. When r = 3 cm the volume increases at 36π cm^3 per second. Find the rate of increase of the radius. [5]

Answer: dV/dr = 4πr^2 = 36π; dV/dt = (dV/dr)(dr/dt), so 36π = 36π(dr/dt) and dr/dt = 1 cm/s.

20. Given y = x + 4/x for x > 0, find the minimum value of y and the corresponding x. [5]

Answer: dy/dx = 1 - 4/x^2 = 0, so x = 2 and y = 2 + 2 = 4; since the second derivative 8/x^3 is positive it is a minimum.

Part 6: Definite Integrals and the Trapezoidal Rule (14 marks)

Give numerical answers to three decimal places where needed.

21. Evaluate the definite integral of 3x^2 + 2x from x = 1 to x = 2. [4]

Answer: Antiderivative x^3 + x^2; (8 + 4) - (1 + 1) = 12 - 2 = 10.

22. Evaluate the definite integral of 1/x from x = 1 to x = e. [4]

Answer: Antiderivative ln x; ln e - ln 1 = 1 - 0 = 1.

23. Use the trapezoidal rule with h = 0.5 and five ordinates to estimate the integral of x^2 from 0 to 2, and state whether the estimate is too large or too small. [3]

Answer: h/2[f(0) + 2(f(0.5) + f(1) + f(1.5)) + f(2)] = 0.25(0 + 7 + 4) = 2.75; the exact value is 8/3 = 2.667, so the estimate is too large.

24. Find the area between the curve y = x^2 and the x-axis from x = 0 to x = 3. [3]

Answer: Area = integral of x^2 = x^3/3 evaluated from 0 to 3, giving 27/3 = 9 square units.
