S5 Mathematics (Extended Part Module 1) Practice Paper | Binomial theorem, exponential and logarithmic functions, differentiation, integration, probability distributions

S5 Mathematics (Extended Part Module 1) | Time allowed: 75 minutes | Total: 100 marks | 24 questions

Question 1 Binomial Theorem

Answer the following questions and show your working.

1. Find the value of C(6, 3). (2 marks)

> **Answer:** C(6, 3) = (6 x 5 x 4) / (3 x 2 x 1) = 20.

2. Find the coefficient of x^3 in the expansion of (1 + 2x)^6. (4 marks)

> **Answer:** Coefficient = C(6, 3) x 2^3 = 20 x 8 = 160.

3. Find the coefficient of x^2 in the expansion of (2 - x)^5. (4 marks)

> **Answer:** Coefficient = C(5, 2) x 2^3 x (-1)^2 = 10 x 8 x 1 = 80.

4. Find the constant term in the expansion of (x + 2/x)^4. (6 marks)

> **Answer:** The (r + 1)th term is C(4, r) x^(4 - r) (2/x)^r = C(4, r) 2^r x^(4 - 2r). Put 4 - 2r = 0, so r = 2. Constant term = C(4, 2) x 2^2 = 6 x 4 = 24.

Question 2 Exponential and Logarithmic Functions

Solve the following equations and state the formula used in the applied part.

5. Solve 3^(2x) = 81. (3 marks)

> **Answer:** 3^(2x) = 3^4, so 2x = 4 and x = 2.

6. Solve log2 x + log2 (x - 2) = 3. (5 marks)

> **Answer:** x(x - 2) = 2^3 = 8, so x^2 - 2x - 8 = 0 and (x - 4)(x + 2) = 0, giving x = 4 or x = -2. Since x > 2, x = -2 is rejected and the answer is x = 4.

7. Simplify (log 27) / (log 3). (3 marks)

> **Answer:** log 27 = log 3^3 = 3 log 3, so the ratio is 3 log 3 / log 3 = 3.

8. A machine worth $50 000 depreciates by 15% each year. Find the least number of years for its value to fall below $20 000. (5 marks)

> **Answer:** Value = 50000 x (0.85)^n < 20000, so (0.85)^n < 0.4. Taking logarithms, n > log 0.4 / log 0.85 = 5.638, hence n = 6. Check: after 5 years $22 185, after 6 years $18 858.

Question 3 Differentiation and Tangents

Differentiate the following functions and answer the tangent question.

9. Let y = (3x + 1)^5. Find dy/dx. (3 marks)

> **Answer:** By the chain rule, dy/dx = 5(3x + 1)^4 x 3 = 15(3x + 1)^4.

10. Let y = x^2 ln x. Find dy/dx. (4 marks)

> **Answer:** By the product rule, dy/dx = 2x ln x + x^2 (1/x) = 2x ln x + x = x(1 + 2 ln x).

11. Let y = (2x - 1)/(x + 3). Find dy/dx. (5 marks)

> **Answer:** By the quotient rule, dy/dx = [2(x + 3) - (2x - 1)] / (x + 3)^2 = 7 / (x + 3)^2.

12. A curve is y = x^3 - 3x. Find the equation of the tangent at x = 2. (6 marks)

> **Answer:** dy/dx = 3x^2 - 3, so at x = 2 the slope is 12 - 3 = 9. The point of contact is y = 8 - 6 = 2. Tangent: y - 2 = 9(x - 2), that is y = 9x - 16.

Question 4 Applications of Differentiation

Find stationary points, extrema and a related rate.

13. A curve is y = x^3 - 6x^2 + 9x. Find all stationary points and determine their nature. (5 marks)

> **Answer:** dy/dx = 3x^2 - 12x + 9 = 3(x - 1)(x - 3), so x = 1 or x = 3. y(1) = 4 and y(3) = 0. d2y/dx2 = 6x - 12: at x = 1 it is -6 < 0, so (1, 4) is a maximum point; at x = 3 it is 6 > 0, so (3, 0) is a minimum point.

14. A rectangular enclosure is made with 40 m of fencing. Find the largest possible area. (5 marks)

> **Answer:** Let the length be x and the width y, then 2x + 2y = 40 and y = 20 - x. Area A = x(20 - x) = 20x - x^2. dA/dx = 20 - 2x = 0 gives x = 10 and y = 10, so the largest area is 10 x 10 = 100 m2.

15. The volume of a sphere is V = (4/3)pi r^3. When r = 5 cm and r increases at 0.2 cm/s, find dV/dt. (3 marks)

> **Answer:** dV/dr = 4 pi r^2, so dV/dt = 4 pi r^2 x dr/dt = 4 pi (25)(0.2) = 20 pi, about 62.8 cm3/s.

16. For the curve y = x^3 - 6x^2 + 9x, find the range of x where the curve is concave upward. (3 marks)

> **Answer:** d2y/dx2 = 6x - 12 > 0 gives x > 2, so the curve is concave upward for x > 2.

Question 5 Integration and Area

Evaluate the following integrals and the area under a curve.

17. Find the integral of (6x^2 - 4x + 3) with respect to x. (3 marks)

> **Answer:** The integral is 2x^3 - 2x^2 + 3x + C.

18. Evaluate the definite integral of (x^3 + 1) from x = 1 to x = 2. (4 marks)

> **Answer:** An antiderivative is x^4/4 + x. Substituting the limits, (16/4 + 2) - (1/4 + 1) = 6 - 1.25 = 4.75, that is 19/4.

19. Find the area bounded by the curve y = x(4 - x) and the x-axis for 0 <= x <= 4. (5 marks)

> **Answer:** Here y = 4x - x^2 >= 0 on the interval, so the area is the integral of 4x - x^2 from 0 to 4, which equals [2x^2 - x^3/3] from 0 to 4 = 32 - 64/3 = 32/3, about 10.7.

20. Find the integral of (2x + 1)^3 with respect to x. (4 marks)

> **Answer:** The integral is (2x + 1)^4 / 8 + C.

Question 6 Binomial and Poisson Distributions

Let X be a binomial random variable and Y a Poisson random variable.

21. Let X be binomial with n = 10 and p = 0.3. Find P(X = 3) correct to 3 significant figures. (5 marks)

> **Answer:** P(X = 3) = C(10, 3)(0.3)^3(0.7)^7 = 120 x 0.027 x 0.0823543 = 0.267.

22. Using the same X, find P(X >= 1) correct to 3 significant figures. (4 marks)

> **Answer:** P(X >= 1) = 1 - P(X = 0) = 1 - (0.7)^10 = 1 - 0.0282475 = 0.972.

23. Find the mean and the variance of X. (4 marks)

> **Answer:** Mean = np = 10 x 0.3 = 3; variance = np(1 - p) = 10 x 0.3 x 0.7 = 2.1.

24. Let Y be Poisson with mean 4. Find P(Y = 0) and P(Y <= 1) correct to 3 significant figures. (5 marks)

> **Answer:** P(Y = 0) = e^(-4) = 0.0183; P(Y <= 1) = e^(-4)(1 + 4) = 0.0916.
