S5 Mathematics Extended Part Module 2 Practice Paper: Matrices and Determinants, Vectors, Binomial Theorem and Mathematical Induction, Calculus of Exponential and Logarithmic Functions, Integration Techniques, Trigonometry and Applications (100 marks)

S5 | Mathematics Extended Part Module 2 | Time: 75 minutes | Total: 100 marks | 24 questions

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Part 1: Matrices and Determinants (15 marks)

Show all working.

1. Let A = [[2, 3], [1, 4]] and B = [[1, -1], [2, 0]]. Find AB. [4]

Answer: Entry (1,1) = 2x1 + 3x2 = 8; (1,2) = 2x(-1) + 3x0 = -2; (2,1) = 1x1 + 4x2 = 9; (2,2) = 1x(-1) + 4x0 = -1. Hence AB = [[8, -2], [9, -1]].

2. Let N = [[k, 2], [3, k + 1]]. Find all values of k for which N is singular (not invertible). [4]

Answer: det N = k(k+1) - 6 = k^2 + k - 6 = (k + 3)(k - 2); setting det N = 0 gives k = -3 or k = 2.

3. Let M = [[4, 1], [2, 3]]. Find det M and the inverse matrix M^(-1). [4]

Answer: det M = 4x3 - 1x2 = 10; M^(-1) = (1/10)[[3, -1], [-2, 4]] = [[3/10, -1/10], [-2/10, 4/10]].

4. Solve the system 3x + y = 5 and 2x + 4y = 10 by a matrix method. [3]

Answer: The determinant of the coefficient matrix is 3x4 - 1x2 = 10, so the solution is unique: x = 1, y = 2. Check: 3(1) + 2 = 5 and 2(1) + 4(2) = 10.

Part 2: Vectors (17 marks)

Give angles to 0.1 degree; keep surd form otherwise.

5. Let a = (3, -1, 2) and b = (1, 4, -2). Find a.b and the angle between a and b (to 0.1 degree). [4]

Answer: a.b = 3 - 4 - 4 = -5; |a| = sqrt(14), |b| = sqrt(21); cos(theta) = -5/sqrt(294) = -0.2916, so theta = 107.0 degrees.

6. Given a = (2, t, 1) and b = (1, -1, 3) are perpendicular, find t. [4]

Answer: Perpendicular means a.b = 0: 2x1 + t x (-1) + 1x3 = 0, i.e. 2 - t + 3 = 0, giving t = 5.

7. Let a = (2, -1, 3) and b = (1, 4, -2). Find a x b. [5]

Answer: a x b = ((-1)(-2) - 3x4, 3x1 - 2x(-2), 2x4 - (-1)(1)) = (-10, 7, 9); the checks (a x b).a = 0 and (a x b).b = 0 both hold.

8. The position vectors of A and B are a = (1, 2, 3) and b = (2, -1, 1). Find the area of triangle OAB. [4]

Answer: a x b = (5, 5, -5) with magnitude sqrt(75) = 5 sqrt(3); the area is half of that, i.e. 5 sqrt(3)/2 = 4.3301 square units.

Part 3: Binomial Theorem and Mathematical Induction (17 marks)

Induction needs the base case, assumption and inductive step.

9. Find the coefficient of x^3 in the expansion of (2 + x)^7. [4]

Answer: Coefficient = C(7,3) x 2^4 = 35 x 16 = 560.

10. Find the term independent of x in the expansion of (x^2 + 2/x)^9. [4]

Answer: The (r+1)th term is C(9,r) 2^r x^(18-3r); setting 18 - 3r = 0 gives r = 6; the term is C(9,6) x 2^6 = 84 x 64 = 5376.

11. Prove by mathematical induction that 1 + 5 + 9 + ... + (4n - 3) = n(2n - 1) for all positive integers n. [4]

Answer: For n = 1, LHS = 1 and RHS = 1x1 = 1, so it holds. Assume it holds for n = m; adding the (m+1)th term (4m+1) gives m(2m-1) + 4m + 1 = 2m^2 + 3m + 1 = (m+1)(2m+1), so it holds for n = m+1. Check n = 5: 1+5+9+13+17 = 45 = 5 x 9.

12. Given that the sum of k(k+1) from k = 1 to n equals n(n+1)(n+2)/3, find the sum from k = 1 to 20. [5]

Answer: Put n = 20: 20 x 21 x 22 / 3 = 3080, which matches a term-by-term check.

Part 4: Exponential and Logarithmic Functions, Differentiation (17 marks)

State the rule used.

13. Let y = x^2 e^(3x). Find dy/dx. [4]

Answer: By the product rule, dy/dx = 2x e^(3x) + x^2 (3 e^(3x)) = e^(3x)(2x + 3x^2).

14. Let y = ln(x^2 + 1). Find dy/dx. [4]

Answer: By the chain rule, dy/dx = (1/(x^2 + 1)) x 2x = 2x/(x^2 + 1).

15. Let y = x ln x for x > 0. Find the equation of the tangent to the curve at x = e. [5]

Answer: y(e) = e ln e = e, so the point is (e, e); dy/dx = ln x + 1 gives slope 2 at x = e; hence the tangent is y - e = 2(x - e), i.e. y = 2x - e.

16. Solve e^(2x) - 3e^x + 2 = 0. [4]

Answer: Let u = e^x > 0: u^2 - 3u + 2 = 0 gives u = 1 or u = 2, so x = 0 or x = ln 2 = 0.6931.

Part 5: Integration (17 marks)

Show the limits substituted for definite integrals.

17. Find the indefinite integral of x e^x with respect to x. [4]

Answer: Integrating by parts: x e^x - e^x + C, i.e. e^x(x - 1) + C.

18. Evaluate the definite integral of ln x from x = 1 to x = e. [4]

Answer: The antiderivative is x ln x - x; at x = e it equals e - e = 0 and at x = 1 it equals 0 - 1 = -1; so the value is 0 - (-1) = 1.

19. Evaluate the definite integral of 4x/(x^2 + 1) from x = 0 to x = 2. [5]

Answer: Since d(x^2 + 1) = 2x dx, the integral equals 2 ln(x^2 + 1) evaluated from 0 to 2, i.e. 2(ln 5 - 0) = 2 ln 5 = 3.2189.

20. Find the area of the region bounded by y = x and y = x^2 between x = 0 and x = 1. [4]

Answer: On 0 <= x <= 1, x >= x^2, so the area is the integral of x - x^2, i.e. 1/2 - 1/3 = 1/6 = 0.1667 square units.

Part 6: Trigonometry and Applications (17 marks)

List every solution in the range 0 <= x < 2 pi.

21. Prove that (sin theta + cos theta)^2 = 1 + sin 2 theta. [4]

Answer: LHS = sin^2(theta) + 2 sin(theta)cos(theta) + cos^2(theta) = 1 + sin(2 theta).

22. Solve 2 sin^2 x - sin x - 1 = 0 for 0 <= x < 2 pi. [5]

Answer: Factorising gives (2 sin x + 1)(sin x - 1) = 0; sin x = 1 gives x = pi/2, while sin x = -1/2 gives x = 7 pi/6 or 11 pi/6. Solutions: pi/2, 7 pi/6, 11 pi/6.

23. Let y = x^3 - 3x^2 + 4. Find the local maximum and local minimum points of the curve. [4]

Answer: dy/dx = 3x^2 - 6x = 3x(x - 2), so the stationary points are x = 0 and x = 2; the local maximum is (0, 4) and the local minimum is (2, 0).

24. The radius r of a circle increases at 2 cm per second. Find the rate of increase of the area A when r = 5 cm. [4]

Answer: A = pi r^2, so dA/dt = 2 pi r (dr/dt); with r = 5 and dr/dt = 2 this gives 20 pi = 62.83 cm^2 per second.
