S6 Mathematics (Extended Part Module 1) Practice Paper

S6 | Mathematics (Extended Part Module 1) | Time: 75 minutes | Total: 100 marks | 24 questions

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Part A Binomial Theorem (16 marks)

Let x be a real number and consider the expansion of (1 + 2x)^10.

1. Find the coefficient of x^3 in the expansion. (4 marks)

Answer: Coefficient of x^3 = C(10,3) x 2^3 = 120 x 8 = 960.

2. Find the coefficient of x^4 in the expansion. (4 marks)

Answer: Coefficient of x^4 = C(10,4) x 2^4 = 210 x 16 = 3360.

3. Find the coefficient of x^2 in the expansion of (1 - 3x)(1 + 2x)^10. (4 marks)

Answer: Coefficient of x^2 = C(10,2) x 2^2 - 3 x C(10,1) x 2 = 180 - 60 = 120.

4. Taking x = 0.01, estimate 1.02^10 using the first three terms and find the percentage error. (4 marks)

Answer: 1 + 10(0.02) + 45(0.02)^2 = 1.218; true value 1.218994; percentage error = 0.0816%.

Part B Exponential and Logarithmic Functions (16 marks)

Give answers to 3 significant figures unless otherwise stated.

5. Solve 5^(2x - 1) = 40. (4 marks)

Answer: 2x - 1 = ln 40 / ln 5 = 2.2920, so x = 1.646.

6. Solve ln(x + 2) + ln(x - 1) = ln 10. (4 marks)

Answer: (x + 2)(x - 1) = 10 gives x^2 + x - 12 = 0, so x = 3 or x = -4, rejected since x > 1. Hence x = 3.

7. A town of population 48000 grows at 3.5% per year. Find the number of years for the population to reach 70000. (4 marks)

Answer: 48000(1.035)^t = 70000, so t = ln(70000 / 48000) / ln 1.035 = 10.97 years, about 11 years.

8. Let y = e^(3x) ln x. Find dy/dx at x = 1. (4 marks)

Answer: dy/dx = e^(3x)(3 ln x + 1/x); at x = 1 this equals e^3 = 20.09.

Part C Differentiation and Its Applications (18 marks)

In the following questions the domain is x > 0 unless otherwise stated.

9. y = x^2 e^(-x). Find all stationary points and determine their nature. (4 marks)

Answer: dy/dx = e^(-x)(2x - x^2), so x = 0 and x = 2. At x = 0, y = 0 is a minimum; at x = 2, y = 4e^-2 = 0.5413 is a maximum.

10. y = ln(x^2 + 1). Find the equation of the tangent at x = 1. (4 marks)

Answer: dy/dx = 2x/(x^2 + 1), so the slope at x = 1 is 1. The point is (1, 0.6931), hence the tangent is y = x - 0.3069.

11. y = (2x + 1)/(x - 1). Find dy/dx and its value at x = 2. (5 marks)

Answer: dy/dx = [2(x - 1) - (2x + 1)]/(x - 1)^2 = -3/(x - 1)^2; at x = 2, dy/dx = -3.

12. The volume of a sphere is V = (4/3) pi r^3 and is increasing at 100 cm^3/s. Find the rate of increase of the radius when r = 5 cm. (5 marks)

Answer: dV/dt = 4 pi r^2 dr/dt, so dr/dt = 100 / (4 pi x 25) = 0.3183 cm/s.

Part D Integration (16 marks)

Evaluate the following integrals, giving answers to 3 significant figures.

13. Evaluate the integral from 0 to 1 of x e^(x^2) dx. (4 marks)

Answer: With u = x^2, du = 2x dx, the integral is (1/2)[e^(x^2)] from 0 to 1 = (e - 1)/2 = 0.8591.

14. Evaluate the integral from 1 to 4 of (3 sqrt(x) + 1/x) dx. (4 marks)

Answer: = [2x^(3/2) + ln x] from 1 to 4 = (16 - 2) + ln 4 = 15.39.

15. Find the integral of (2x + 3)/(x^2 + 3x + 5) dx and evaluate it from 0 to 1. (4 marks)

Answer: = ln(x^2 + 3x + 5) + C; the definite integral is ln(9/5) = 0.5878.

16. Find the area bounded by y = x(4 - x) and the x-axis for 0 <= x <= 4. (4 marks)

Answer: Area = integral from 0 to 4 of (4x - x^2) dx = [2x^2 - x^3/3] = 32/3 = 10.67.

Part E Probability Distributions (18 marks)

Let X ~ B(20, 0.3) and Y ~ Po(4). The parts below are independent.

17. Find P(X = 6), E(X) and Var(X). (4 marks)

Answer: P(X = 6) = C(20,6)(0.3)^6(0.7)^14 = 0.1916; E(X) = 20 x 0.3 = 6; Var(X) = 20 x 0.3 x 0.7 = 4.2.

18. Find P(Y <= 2). (4 marks)

Answer: P(Y <= 2) = e^-4 (1 + 4 + 8) = 0.2381.

19. Let Z ~ B(200, 0.4). Use a normal approximation with continuity correction to find P(Z >= 90). (5 marks)

Answer: mu = 80, sigma = sqrt(48) = 6.928, z = (89.5 - 80)/sigma = 1.37, so P(Z >= 90) is about 0.0852.

20. A process has a 3% defective rate. A sample of 100 items is taken. Using a Poisson approximation, find the probability of at most 1 defective item. (5 marks)

Answer: lambda = 100 x 0.03 = 3, so P(W <= 1) = e^-3 (1 + 3) = 0.1991.

Part F Statistical Inference (16 marks)

Give answers to 3 significant figures unless otherwise stated.

21. A sample of size 9 has mean 24.6 and standard deviation 3.0. Find the 95% confidence interval for the population mean. (4 marks)

Answer: 24.6 +/- 1.96 x 3.0/sqrt(9) = 24.6 +/- 1.96, that is (22.64, 26.56).

22. Using the same setting with standard deviation about 5.0, find the minimum sample size so that the half-width of the interval does not exceed 1.0. (4 marks)

Answer: n >= (1.96 x 5.0 / 1.0)^2 = 96.04, so the minimum sample size is 97.

23. For the data 4, 7, 9, 12, 13, find the mean, the unbiased estimate of the variance and the sample standard deviation. (4 marks)

Answer: Mean = 9; sum of squared deviations = 54; s^2 = 13.5; s = 3.674.

24. A sample of size 100 has mean 52 and standard deviation 8. Test H0: mu = 50 at the 5% level using a two-tailed test. (4 marks)

Answer: z = (52 - 50) / (8/10) = 2.5 and the p-value is 0.0124 < 0.05, so reject H0.
