S6 Mathematics Module 2 Practice Paper 1 | limits, differentiation, integration, matrices, vectors, binomial, optimisation

Senior Secondary Mathematics

S6 Mathematics Module 2 practice paper with worked solutions: limits, differentiation, integration, matrices, vectors, binomial distribution, optimisation.

1. Limits and differentiation techniques (18 marks)

Let f(x) = e^(2x) ln x, let C be the curve x^2 + y^2 = 25, and let g(x) = x^3 - 3x.

#	Question	Answer and marking notes
1	(a) Find lim(x->0) (sin 3x) / x.	Answer: as x tends to 0, sin 3x is approximately 3x, so (sin 3x)/x tends to 3; equivalently 3 x lim(sin 3x)/(3x) = 3 x 1 = 3.
2	(b) Find f'(x) and the exact value of f'(1).	Answer: by the product rule f'(x) = 2e^(2x) ln x + e^(2x)/x; at x = 1, f'(1) = 2e + e = 3e, about 8.15.
3	(c) At the point (3, 4) on C, find dy/dx.	Answer: differentiating implicitly gives 2x + 2y(dy/dx) = 0, so dy/dx = -x/y; at (3, 4), dy/dx = -3/4 = -0.75.
4	(d) Find the maximum and minimum points of g(x) and determine their nature.	Answer: g'(x) = 3x^2 - 3 = 0 gives x = 1 or x = -1; g”(x) = 6x, so g”(1) = 6 > 0 giving a minimum with g(1) = -2, and g”(-1) = -6 is less than 0 giving a maximum with g(-1) = 2.

2. Integration techniques and volume of revolution (16 marks)

Evaluate the following, giving answers correct to three decimal places where appropriate.

#	Question	Answer and marking notes
5	(a) Find the exact value of the integral of (3x^2 + 2x) dx from 1 to 2.	Answer: the antiderivative is x^3 + x^2, so the value is (8 + 4) - (1 + 1) = 10.
6	(b) Use integration by parts to find the exact value of the integral of x e^x dx from 0 to 1.	Answer: take u = x and dv = e^x dx, so du = dx and v = e^x; the integral equals [x e^x] from 0 to 1 minus the integral of e^x from 0 to 1, giving e - (e - 1) = 1.
7	(c) Use substitution to find the integral of 2x/(x^2 + 1) dx from 0 to 1, correct to three decimal places.	Answer: let u = x^2 + 1 with du = 2x dx, so the integral is ln(x^2 + 1) from 0 to 1 = ln 2 - ln 1 = ln 2, about 0.693.
8	(d) The curve y = sqrt(x) from x = 0 to x = 4 is rotated about the x-axis. Find the volume of the solid in terms of pi.	Answer: V = pi times the integral of x from 0 to 4, which is pi[x^2/2] from 0 to 4 = 8 pi, about 25.133 cubic units.

3. Matrices and systems of linear equations (18 marks)

Let A be the 2 by 2 matrix with rows (2, 1) and (5, 3). Consider the system 2x + y = 7 and 5x + 3y = 18.

#	Question	Answer and marking notes
9	(a) Find det A and the inverse of A.	Answer: det A = 2 x 3 - 1 x 5 = 1, so the inverse is (1/1) times the matrix with rows (3, -1) and (-5, 2), that is rows (3, -1) and (-5, 2).
10	(b) Use the inverse matrix to solve the system.	Answer: (x, y) equals the inverse times (7, 18), so x = 3 x 7 - 1 x 18 = 3 and y = -5 x 7 + 2 x 18 = 1; thus x = 3 and y = 1.
11	(c) Evaluate the determinant of the 3 by 3 matrix with rows (1, 2, 3), (0, 1, 4) and (5, 6, 0).	Answer: expanding along the first row gives 1(0 - 24) - 2(0 - 20) + 3(0 - 5) = -24 + 40 - 15 = 1.
12	(d) If the determinant of a 3 by 3 matrix is 0, what can be concluded about the corresponding system of equations?	Answer: the matrix has no inverse, so the system has no unique solution; it is either inconsistent or has infinitely many solutions, and consistency must be checked separately.

4. Dot products, cross products and areas (18 marks)

Let vector a be (3, -1, 2) and vector b be (1, 4, -2).

#	Question	Answer and marking notes
13	(a) Find the dot product of a and b.	Answer: the dot product is 3 x 1 + (-1) x 4 + 2 x (-2) = 3 - 4 - 4 = -5.
14	(b) Find the angle between a and b, correct to one decimal place.	Answer: the magnitudes are sqrt 14, about 3.742, and sqrt 21, about 4.583, so cos theta = -5 / sqrt 294, about -0.2916, giving theta about 107.0 degrees.
15	(c) Find the cross product of a and b.	Answer: the cross product is ((-1)(-2) - 2 x 4, 2 x 1 - 3(-2), 3 x 4 - (-1)(1)) = (-6, 8, 13).
16	(d) Find the area of the parallelogram with a and b as adjacent sides, correct to two decimal places.	Answer: the area is the magnitude of the cross product, sqrt(36 + 64 + 169) = sqrt 269, about 16.40 square units.

5. The binomial distribution (16 marks)

The defective rate of components from a factory is 5 per cent. Twenty components are chosen at random and X, the number of defectives, follows a binomial distribution.

#	Question	Answer and marking notes
17	(a) Find the probability that X equals 0, correct to four decimal places.	Answer: the probability is 0.95 to the power 20, about 0.3585.
18	(b) Find the probability that X is at least 1, correct to four decimal places.	Answer: the probability is 1 minus 0.3585, which is 0.6415.

#	Question	Answer and marking notes
19	(c) Find the mean, variance and standard deviation of X, giving the standard deviation to four decimal places.	Answer: the mean is np = 20 x 0.05 = 1; the variance is np(1 - p) = 0.95; the standard deviation is the square root of 0.95, about 0.9747.
20	(d) Find the probability that X equals 2, correct to four decimal places.	Answer: the probability is C(20, 2) times 0.05 squared times 0.95 to the power 18, that is 190 x 0.0025 x 0.3972, about 0.1887.

6. Optimising the surface area of an open tank (16 marks)

An open-top rectangular tank has a square base of side x metres and a fixed capacity of 32 cubic metres. Its height is h metres.

#	Question	Answer and marking notes
21	(a) Express h in terms of x and write down the surface area A.	Answer: from x^2 h = 32, h = 32/x^2; the surface area is x^2 + 4xh = x^2 + 128/x square metres.
22	(b) Find the value of x that minimises A and the corresponding h.	Answer: setting the derivative 2x - 128/x^2 to zero gives x^3 = 64, so x = 4 and h = 32/16 = 2 metres.
23	(c) Verify that this value of x gives a minimum and find the minimum surface area.	Answer: the second derivative 2 + 256/x^3 equals 6 at x = 4, which is positive, so it is a minimum; the area is 16 + 32 = 48 square metres.
24	(d) If the material costs 150 dollars per square metre, find the material cost at the minimum surface area, and the balance left from a budget of 8000 dollars.	Answer: the cost is 48 x 150 = 7200 dollars, leaving 8000 - 7200 = 800 dollars.