S4 Mathematics Extended Module 2 Practice Paper: Mathematical Induction, Binomial Theorem, Compound Angles and Trigonometric Equations, Vectors, Matrices and Linear Transformations, Limits, Derivatives and Integration

S4 | Mathematics Extended Part (Module 2) | Time: 75 minutes | Total: 100 marks | 24 questions

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Part 1: Mathematical Induction (15 marks)

State the base case, the induction hypothesis and the induction step clearly.

1. Prove by mathematical induction that 1 + 3 + 5 + ... + (2n - 1) = n^2 for all positive integers n. [3]

Answer: n = 1: LHS = 1 = 1^2. Assume true for n = k, i.e. the sum of the first k odd numbers is k^2. For n = k + 1 the sum is k^2 + (2k + 1) = (k + 1)^2, so the statement holds for all positive integers n.

2. Prove that 2^n > n^2 for all integers n >= 5. [4]

Answer: n = 5: 2^5 = 32 > 25 = 5^2. Assume 2^k > k^2 with k >= 5. Then 2^(k+1) = 2 x 2^k > 2k^2, and 2k^2 - (k + 1)^2 = k^2 - 2k - 1 > 0 for k >= 5, so 2^(k+1) > (k + 1)^2.

3. Prove that the sum of r x 2^r from r = 1 to n equals (n - 1)2^(n+1) + 2 for all positive integers n. [4]

Answer: n = 1: LHS = 2, RHS = 0 + 2 = 2. Assume true for n = k. Then the sum to k + 1 is (k - 1)2^(k+1) + 2 + (k + 1)2^(k+1) = 2k(2^(k+1)) + 2 = k(2^(k+2)) + 2, which is the formula with n = k + 1.

4. Prove that 7^n - 1 is divisible by 6 for all positive integers n. [4]

Answer: n = 1: 7 - 1 = 6. Assume 7^k - 1 = 6M for an integer M. Then 7^(k+1) - 1 = 7(6M + 1) - 1 = 42M + 6 = 6(7M + 1), again a multiple of 6.

Part 2: Binomial Theorem (17 marks)

Simplify every term; leave coefficients as integers or fractions in lowest terms.

5. Expand (2x - 3)^4. [5]

Answer: 16x^4 - 96x^3 + 216x^2 - 216x + 81.

6. Find the coefficient of x^6 in the expansion of (x^2 + 2/x)^6. [4]

Answer: The general term is C(6, r)(x^2)^(6-r)(2/x)^r = C(6, r)2^r x^(12-3r). Set 12 - 3r = 6, so r = 2 and the coefficient is 15 x 4 = 60.

7. Find the constant term in the expansion of (2x - 1/(2x))^8. [4]

Answer: The general term is C(8, r)(2x)^(8-r)(-1/(2x))^r = C(8, r)2^(8-2r)(-1)^r x^(8-2r). Setting 8 - 2r = 0 gives r = 4, so the constant term is 70.

8. Given that the coefficients of x and x^2 in the expansion of (1 + kx)^n are 24 and 252 respectively, find n and k. [4]

Answer: C(n,1)k = 24 gives nk = 24; C(n,2)k^2 = 252 gives n(n - 1)k^2/2 = 252. Using nk = 24 gives 12k(n - 1) = 252, so k(n - 1) = 21. Together with nk = 24 this yields n = 8 and k = 3.

Part 3: Compound Angles and Trigonometric Equations (17 marks)

Unless stated otherwise, give angles in degrees correct to 3 significant figures.

9. Given sin A = 3/5 and cos B = 5/13, where A and B are acute, find the exact value of sin(A + B). [4]

Answer: cos A = 4/5 and sin B = 12/13, so sin(A + B) = sin A cos B + cos A sin B = 15/65 + 48/65 = 63/65.

10. Solve 2sin^2 θ + sin θ - 1 = 0 for 0 <= θ < 360 degrees. [4]

Answer: Factorising gives (2 sin θ - 1)(sin θ + 1) = 0, so sin θ = 1/2 or sin θ = -1, giving θ = 30, 150 or 270 degrees.

11. Simplify sin 2θ / (1 + cos 2θ) and state the restriction on θ. [4]

Answer: sin 2θ = 2 sin θ cos θ and 1 + cos 2θ = 2 cos^2 θ, so the quotient is tan θ, provided cos θ is not zero, i.e. θ is not 90 + 180n degrees.

12. In triangle ABC, a = 7, b = 9 and C = 60 degrees. Find c and the area of the triangle, correct to 3 significant figures. [5]

Answer: By the cosine formula, c^2 = 49 + 81 - 2(7)(9)cos 60 = 130 - 63 = 67, so c = 8.19. Area = 0.5 x 7 x 9 x sin 60 = 27.3.

Part 4: Vectors (17 marks)

Write vectors as ordered pairs; give magnitudes as exact values or 3 significant figures.

13. Given a = (3, -1) and b = (-2, 4), find 2a - b, |2a - b| and the unit vector in the direction of 2a - b. [4]

Answer: 2a - b = (8, -6), |2a - b| = sqrt(64 + 36) = 10, and the unit vector is (0.8, -0.6).

14. Given that a = (1, 2) and b = (k, 3) are perpendicular, find k. [4]

Answer: Perpendicular vectors have zero dot product: k + 6 = 0, so k = -6.

15. Find the angle between the vectors (1, 1) and (1, 0). [4]

Answer: cos θ = 1/(sqrt2 x 1) = 1/sqrt2, so the angle is 45 degrees.

16. Given A(1, 2), B(4, 6) and C(7, 2), find the area of triangle ABC. [5]

Answer: AB = (3, 4) and AC = (6, 0), so the area is 0.5 |3 x 0 - 4 x 6| = 12 square units.

Part 5: Matrices and Linear Transformations (17 marks)

Show the multiplication step by step; do not skip working by using a calculator matrix function.

17. Let A = [[2, 1], [3, 4]] and B = [[1, 0], [2, 1]]. Find AB and BA, and hence explain why matrix multiplication is not commutative. [4]

Answer: AB = [[4, 1], [11, 4]] and BA = [[2, 1], [7, 6]]; since they differ, AB is not equal to BA.

18. Find the inverse of the matrix [[3, 1], [2, 4]]. [4]

Answer: The determinant is 12 - 2 = 10, so the inverse is (1/10)[[4, -1], [-2, 3]].

19. Solve the system 2x + y = 7 and x - y = 2 by the matrix method. [4]

Answer: The coefficient matrix has determinant -2 - 1 = -3; its inverse times (7, 2) gives x = 3 and y = 1.

20. The matrix M = [[0, -1], [1, 0]] represents an anticlockwise rotation of 90 degrees. Find the image of (3, 2) under M and state the transformation represented by M^2. [5]

Answer: M(3, 2) = (-2, 3), so the image is (-2, 3). M^2 = [[-1, 0], [0, -1]], a rotation of 180 degrees about the origin.

Part 6: Limits, Derivatives and Integration (17 marks)

Show the simplification for limits; add the constant C for indefinite integrals.

21. Evaluate lim (x -> 3) (x^2 - 9)/(x - 3). [4]

Answer: (x^2 - 9)/(x - 3) = x + 3 for x not equal to 3, so the limit is 6.

22. Find d/dx of x^2 ln x. [4]

Answer: By the product rule the derivative is 2x ln x + x^2(1/x) = 2x ln x + x.

23. Evaluate the integral from 1 to 2 of (3x^2 + 2/x) dx, giving the exact value. [4]

Answer: The antiderivative is x^3 + 2 ln x, so the value is (8 + 2 ln 2) - 1 = 7 + 2 ln 2, about 8.39.

24. Find the equation of the tangent to the curve y = x^3 - 2x at x = 1. [5]

Answer: y prime = 3x^2 - 2, so the slope at x = 1 is 1 and the point is (1, -1). The tangent is y + 1 = x - 1, that is y = x - 2.
